Int. J. Heat Mass Transfer. Vol. 8, pp. 1217–1234. Pergamon Press 1965. Printed in Great Britain

AN APPROXIMATE ANALYTICAL DERIVATION OF SKIN FRICTION AND HEAT TRANSFER IN LAMINAR BINARY BOUNDARY LAYER FLOW†

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(*Received* 30 *July* 1964 *and in revised form* 5 *February* 1965)

Abstract—This paper presents an approximate analysis of the binary boundary-layer equations under conditions in which the external flow field pressure gradient is not zero, and the flow Mach number is not necessarily small. Expressions for the skin friction and heat-transfer coefficients are derived together with formulae exhibiting the explicit effects of injection. The results are compared with exact numerical solutions for a wide range of flow conditions, and the agreement is very close.

NOMENCLATURE

A, skin-friction parameter defined by equation (41);
A_2, constant defined by equation (25);
A_3, constant defined by equation (30);
B, mass-transfer parameter defined by equation (50);
B', mass-transfer parameter defined by equation (41);
$c = \frac{\rho\mu}{\rho_e\mu_e}$, density viscosity parameter;
C^*, Chapman–Rubesin parameter;
C_f, skin-friction coefficient;
C_H, Stanton number;
C_{H_c}, Stanton number based on conduction heat transfer;
C_p, specific heat at constant pressure for mixture;
C_{pF}, specific heat of frozen mixture at constant pressure;
C_{pi}, specific heat at constant pressure for i component;
D_{12}, coefficient of diffusion;
f, modified stream function;
f_w, surface mass-transfer parameter;
$\bar{f}_w = \frac{\rho_w v_w}{\rho_e\mu_e}(R_{ex})^{\frac{1}{2}}$, modified surface mass-transfer parameter;
F^*, function defined by equation (73);
$F(\eta)$, function defined by equation (35);
$g = \frac{H}{H_e}$, total enthalpy ratio;
G, function defined by equation (92);
h, specific enthalpy of mixture;
h_i, specific enthalpy of i component;
H, total enthalpy of mixture;
K_i, mass fraction of i component;
k, thermal conductivity;
$L = \frac{C_{pF}\rho D_{12}}{k}$, Lewis number;
M_e, free stream Mach number;
$\mathscr{M}_i$, molecular weight of i component;
$\bar{\mathscr{M}}$, mean molecular weight of mixture;
p, static pressure;
$P = \frac{\mu C_{pF}}{k}$, Prandtl number;
q, local heat-transfer rate;
q_w, surface heat-transfer rate;
$R_{ex} = \frac{\rho_e u_e x}{\mu_e}$, Reynolds number based on free stream quantities;
R_u, universal gas constant;
$\bar{R}$, mean gas constant of mixture;
$S = \frac{\mu}{\rho D_{12}}$, Schmidt number;
T, temperature;

† This work was sponsored by the Office of Aerospace Research, Aerospace Research Laboratories, under Contract AF 33(616) 8453.

u, v, velocity components in x and y directions;
$\bar{V}_i$, diffusion velocity vector of i component;
x, y, boundary-layer coordinates.

Greek symbols
α, mass-transfer parameter defined by equation (53);
β, pressure gradient parameter;
$\bar{\beta}$, pressure gradient parameter for $M_e = 0$;
γ, ratio of specific heats of free stream gas;
ϵ $= \dfrac{\mathscr{M}_2 C_{p2}}{\mathscr{M}_1 C_{p1}}$, defined by equation (22);
φ, defined by equation (34);
ψ, stream function;
η, ξ, transformed coordinates;
τ_w, wall shear stress;
μ, coefficient of viscosity.

Subscripts
1, refers to foreign gas injected at surface;
2, refers to air in free stream;
e, evaluated in the free stream;
w, evaluated at the wall;
0, evaluated under conditions in which the wall mass transfer is zero, but the wall temperature is maintained constant;
wAD, evaluated under adiabatic wall conditions.

INTRODUCTION

The problem of mass-transfer cooling of vehicles moving at high velocities has been considered by several authors. The general approach, however, has been to solve the complex differential equations by numerical integration on a digital computer. This method, while it provides solutions of very high accuracy, only provides solutions in numerical form. This drawback is very great, particularly from the designer's point of view, since in any specific problem the complete integration must be carried out. For most practical configurations the procedure becomes exceedingly complex, and for problems involving the prediction of surface temperature variations during a reentry phase, this is particularly true.

It was on consideration of these difficulties that a paper was published by Li [1] in which a method of approximate solution of the binary boundary-layer equations was presented. Li's paper, however, dealt only with the case of the flow over a flat plate. The purpose of the present paper is to extend the approach so that it may be applied to the general case with a non-zero pressure gradient. The specific objective is to provide formulas for the evaluation of the skin-friction and heat-transfer coefficients, so that the effects of injection may be observed. The results of the analysis were compared with exact solutions and showed very good agreement. However, for the case where neither the Mach number nor the pressure gradient was zero, exact solution was not available, and so the accuracy of the present method is not known.

The present method of solution is immediately applicable to more complicated problems such as vectorial injection; this may therefore serve as a possible extension. Also, the derived expressions for the heat transfer may be used in numerical procedures to provide relatively simple solutions to the transient heating of a high-speed vehicle.

ANALYSIS

The equations of flow for a binary boundary layer over a surface with non-zero pressure gradient are:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad \text{continuity equation} \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad \text{momentum equation} \tag{2}$$

$$\rho u\, C_{pF} \frac{\partial T}{\partial x} + \rho v\, C_{pF} \frac{\partial T}{\partial y} - \mu \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \rho D_{12}(C_{p1} - C_{p2}) \frac{\partial T}{\partial y} \frac{\partial K_1}{\partial y} + u \frac{\partial p}{\partial x} \quad \text{energy equation} \tag{3}$$

$$\rho u \frac{\partial K_i}{\partial x} + \rho v \frac{\partial K_i}{\partial y} = \frac{\partial}{\partial y}\left(\rho D_{12} \frac{\partial K_i}{\partial y}\right) \quad \text{diffusion equation} \tag{4}$$

where $$\sum_{i=1}^{2} K_i = 1$$

These equations have been derived by making use of the usual boundary-layer approximations. The diffusion equation was derived assuming thermal and pressure diffusions to be negligible relative to mass diffusion.

The boundary conditions on this set of equations are:

$$y = 0, \quad u = 0, \quad T = T_w, \quad K_1 = K_{1w} \tag{5}$$

$$(\rho v)_w = -\frac{\rho_w D_{12}}{1 - K_{1w}} \left(\frac{\partial K_1}{\partial y}\right)_w \tag{6}$$

$$y \to \infty, \quad u \to u_e, \quad T \to T_e, \quad K_1 \to 0. \tag{7}$$

Equation (6) states that the overall velocity at the surface is equal to the relative diffusion velocity. Equations (5) and (7) are the conventional conditions usually used in the integration of boundary-layer equations. The momentum and energy equations may be combined to yield a modified energy equation:

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{P}\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial y}\left[\mu \frac{(P-1)}{P}\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right] + \frac{\partial}{\partial y}\left[\frac{L-1}{L}\rho D_{12} \sum_{i=1}^{2} h_i \frac{\partial K_i}{\partial y}\right] \tag{8}$$

It is now desired to reduce equations (2), (8), (4) to total differential equations by assuming similar solutions.

The transformation equations necessary are:

$$\eta = \frac{\rho_e u_e}{(2\xi)^{\frac{1}{2}}} \int_0^y \frac{\rho}{\rho_e} \mathrm{d}y \tag{9}$$

$$\xi = \int_0^x \rho_e u_e \mu_e \, \mathrm{d}x \tag{10}$$

We may define a function $\psi(x, y)$ to be the solution of the continuity equation, and thus,

$$\frac{\partial \psi}{\partial y} = \rho u \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\rho v \tag{11}$$

In terms of the transformed variables, we may express ψ in a non-dimensional form,

$$\psi(\xi, \eta) = (2\xi)^{\frac{1}{2}} f(\eta) \tag{12}$$

and hence from equations (11) and (12)

$$\frac{u}{u_e} = f'(\eta) \tag{13}$$

Let us introduce the solutions of the energy equation and diffusion equation in similar forms:

$$\frac{H}{H_e} = g(\eta) \tag{14}$$

$$\text{and } K_i = K_i(\eta) \tag{15}$$

Thus, by introduction of the variables η and ξ, and $f(\eta)$, $g(\eta)$, $K_i(\eta)$ we may obtain the following forms of equations (2), (8) and (4):

$$(cf'')' + ff'' = \frac{2\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi}\left(f'^2 - \frac{\rho_e}{\rho}\right) \tag{16}$$

$$\left(\frac{c}{P}g'\right)' + fg' = \frac{2\xi}{H_e} f' g \frac{\mathrm{d}H_e}{\mathrm{d}\xi} + \left[\frac{u_e^2}{2H_e}\left(1 - \frac{1}{P}\right)c(f'^2)'\right]' + \left[\frac{c}{S}\left(\frac{1}{L} - 1\right)\sum_{i=1}^{2}\frac{h_i K_i'}{H_e}\right]' \tag{17}$$

$$\left(\frac{c}{S}K_i'\right)' + fK_i' = 0 \tag{18}$$

where $c = \dfrac{\rho\mu}{\rho_e \mu_e}$ is the density–viscosity ratio.

The transformed boundary conditions become:

$$\left.\begin{aligned} &\eta = 0, f' = 0, g = \frac{H_w}{H_e}, K_1 = K_{1w}, \\ &f = \frac{c_w (K_1')_w}{S_w K_{2w}} \\ &\eta \to \infty, f' \to 1, g \to 1, K_1 \to 0 \end{aligned}\right\} \tag{19}$$

Since we have assumed similar solutions, it is necessary that all coefficients in equations (16) and (17) should be independent of ξ. Thus, the term $(\mathrm{d}H_e/\mathrm{d}\xi)$ in equation (17) must be taken as zero, or the quantity $\frac{2\xi}{H_e}\frac{\mathrm{d}H_e}{\mathrm{d}\xi}$ must be constant. However, if we assume constant total enthalpy in the external flow, then the first condition is satisfied.

If we use the assumption that the pressure variation across the boundary layer is negligible, then we may obtain the density ratio:

$$\frac{\rho_e}{\rho} = \frac{\mathscr{M}_e}{\bar{\mathscr{M}}}\frac{C_{pe}}{C_p}\left[g + \frac{\gamma_e - 1}{2}M_e^2(g - f'^2)\right] \quad (20)$$

According to the simple kinetic theory, we have

$$C_{pi} = \frac{j_i + 2}{2}\frac{R_u}{\mathscr{M}_i} \quad (21)$$

where j_i = the effective number of degrees of freedom of the i gas. It follows that

$$\epsilon = \frac{C_{pe}\mathscr{M}_e}{C_p\bar{\mathscr{M}}} \approx \frac{C_{pe}\mathscr{M}_e}{C_{pF}\bar{\mathscr{M}}} = \frac{j_e + 2}{j_M + 2} \quad (22)$$

where

$$[(j_M + 2)/2] = \{\sum_{i=1}^{2} [K_i/\mathscr{M}_i]\,[(j_i + 2)/2]\}/\{\sum_{i=1}^{2} (K_i/\mathscr{M}_i)\}.$$

Thus ϵ, as defined in equation (22), is a function of the mixture composition. For hydrogen–air mixtures we take $\epsilon = 1$. In the case of helium–air mixtures in which the helium concentration is small, we may take $\epsilon = 1$.

Thus equation (16) may be written:

$$(cf'')' + ff'' + \frac{2\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi}\left(1 + \frac{\gamma_e - 1}{2}M_e^2\right)(g - f'^2) = 0. \quad (23)$$

Let

$$\frac{2\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi}\left(1 + \frac{\gamma_e - 1}{2}M_e^2\right) = \frac{2\xi}{M_e}\frac{\mathrm{d}M_e}{\mathrm{d}\xi} = \beta \quad (24)$$

Thus, for β constant:

$$M_e = A_2\,(\xi)^{\beta/2} \quad (25)$$

This relation gives the external Mach number distribution for similar solutions.

Thus for non-vanishing Mach number in the external stream, the equations are:

$$(cf'')' + ff'' + \beta(g - f'^2) = 0 \quad (26)$$

$$\left(\frac{c}{P}g'\right)' + fg' = \frac{1}{H_e}\left[\frac{c}{S}\left(\frac{1}{L} - 1\right)\sum_{i=1}^{2} h_i K_i'\right]' - \left[\frac{u_e^2}{H_e}\left(1 - \frac{1}{P}\right)cf'f''\right]' \quad (27)$$

$$\left(\frac{c}{S}K_i'\right)' + fK_i' = 0 \quad (28)$$

These equations must be integrated with the boundary conditions as given in equation (19). Clearly, equation (27) cannot have similar solutions unless the term u_e^2/H_e is either zero or a constant, or if $P = 1$.

In the case $u_e^2/H_e = 0$, (or $M_e = 0$) from equation (24) we have:

$$\frac{2\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi} = \beta \quad (29)$$

Thus, for $\bar{\beta}$ constant

$$u_e = A_3\,(\xi)^{\bar{\beta}/2} \quad (30)$$

which gives the external velocity distribution for similar solutions.

The three equations to be solved, for $M_e = 0$, are:

$$(cf'')' + ff'' + \bar{\beta}\,(g - f'^2) = 0 \quad (31)$$

$$\left(\frac{c}{P}g'\right)' + fg' = \frac{1}{H_e}\left[\frac{c}{S}\left(\frac{1}{L} - 1\right)\sum_{i=1}^{2} h_i K_i'\right]' \quad (32)$$

$$\left(\frac{c}{S}K_i'\right)' + fK_i' = 0 \quad (33)$$

METHOD OF SOLUTION

The method of solution adopted is similar to that used by Li in [1]. The method is also based on an analysis of the laminar boundary-layer equations by Meksyn [2].

It is necessary to solve the set of nonlinear differential equations (26), (27), (28) with the given boundary conditions. The solution is based on the assumption that $f''(\eta)$ being a rapidly varying function may be expressed in the form:

$$f''(\eta) = \exp\,[-F(\eta)]\,\varphi(\eta) \tag{34}$$

where

$$F(\eta) = \int_0^{\eta} \frac{f(\eta)}{c}\,\mathrm{d}\eta \tag{35}$$

First, however, we must obtain $f(\eta)$ expressed in the form of a power series in η which satisfies the boundary conditions. Clearly, this may be done by substitution of an assumed series for $f(\eta)$ into equation (26). It can be seen that the function $g(\eta)$ also appears in equation (26), and so to obtain a first approximation we assume Crocco's integral to be valid, i.e.

$$g(\eta) = g_w + f'(\eta)\,(1 - g_w) \tag{36}$$

Surely this expression is only valid for $P = 1$, $L = 1$ and $\beta = 0$. However, so long as $g(\eta)$ may be expressed in the polynomial form

$$g(\eta) = g_w + \sum_{n=1}^{\infty} \frac{c_n \eta^n}{n!} \tag{37}$$

Then to the order of accuracy of the present theory, the solution is not affected by the substitution of Crocco's integral.

Having obtained a power series expansion of $f(\eta)$, we now proceed to substitute this into equation (26) which may be regarded as a first-order linear non-homogeneous equation in $f''(\eta)$. Thus, $f''(\eta)$ can be obtained in the form of equation (34).

Knowing $f''(\eta)$ we may then integrate the expression and apply the boundary conditions at infinity to determine a single unknown coefficient.

Solution of equation (26)

Assume

$$f(\eta) = \sum_{n=0}^{\infty} \frac{a_n \eta^n}{n!} \tag{38}$$

Using the boundary condition $f'(\eta) = 0$ at $\eta = 0$ it follows that

$$a_1 = 0.$$

Similarly, using equation (19),

$$a_0 = \frac{K_1'(o)}{K_{2w}} \frac{c_w}{S_w} \tag{39}$$

Thus, let us take:

$$f(\eta) = \frac{K_1'(o)}{K_{2w}} \frac{c_w}{S_w} + \tfrac{1}{2} f''(o)\,\eta^2 + \sum_{n=3}^{\infty} \frac{a_n \eta^n}{n!} \tag{40}$$

Let

$$\frac{K_1'(o)}{K_{2w}} \frac{c_w}{S_w} = -B'$$

and

$$\frac{f''(o)}{6c_w} = A \tag{41}$$

Since we are mainly interested in quantities at the surface ($\eta = 0$), it is permissible to take $c = c_w$ in equation (26). Thus, substituting equations (38) and (36) into equation (26), and equating the coefficients of η^n in the resulting equation to zero, we obtain:

$$\frac{c_w\, a_{n+3}}{N!} + \beta(1 - g_w)\frac{a_{n+1}}{N!} + i\beta g_w + \sum_{n=0}^{\infty} \frac{a_n}{n!} \frac{a_{N-n+2}}{(N-n)!} - \beta \sum_{n=0}^{N} \frac{a_{n+1}\; a_{N-n+1}}{(n+1)!\,(N-n)!} = 0 \tag{42}$$

where

$$i = 0 \;\text{ for }\; N \neq 0$$
$$i = 1 \;\text{ for }\; N = 0$$

Thus, we may obtain any coefficient, a_n, in terms of a_0 and a_2. However, a_2 remains undetermined even though a_1 and a_0 are known. From the form of the equation, the reason for this is clear.

Computations using equation (42) give the following results for the coefficients.

$$a_0 = -B', \quad a_1 = 0, \quad a_2 = f''(o) = a,$$

$$a_3 = \frac{B'a - \beta g_w}{c_w},$$

$$a_4 = -\frac{\beta}{c_w}(1 - g_w)\, a + \frac{B'^2 a - \beta g_w B'}{c_w^2} \qquad (43)$$

Taking only the first two terms in $f(\eta)$, we may expand the exponential in equation (34) to give:

$$\exp[-F(\eta)] = 1 + \frac{B'}{c_w}\eta + \frac{B'^2}{2c_w^2}\eta^2 + \left(\frac{B'^3}{6c_w^3} - \frac{a}{6c_w}\right)\eta^3 + \ldots \qquad (44)$$

Let us take

$$\varphi(\eta) = \sum_{n=0}^{\infty} \frac{b_n \eta^n}{n!} \qquad (45)$$

Then

$$f''(\eta) = b_0 + \eta\left(b_0\frac{B'}{c_w} + b_1\right) + \eta^2\left(\frac{b_0 B'^2}{2c_w^2} + \frac{b_1 B'}{c_w} + \frac{b_2}{2}\right) \qquad (46)$$

However, from equations (40) and (43) we have

$$f''(\eta) = a - \frac{a_0 a + \beta g_w}{c_w}\eta + \left[-\beta(1 - g_w)\frac{a}{c_w} + \frac{B'^2 a - B'\beta g_w}{c_w^2}\right]\frac{\eta^2}{2} + o(\eta^3) \qquad (47)$$

Comparing equations (47) and (46) yields

$$b_0 = a, \quad b_1 = -\frac{\beta g_w}{c_w},$$

$$b_2 = \frac{B'\beta g_w}{c_w^2} - \frac{\beta a}{c_w}(1 - g_w) \qquad (48)$$

Thus, by equation (45), we have

$$\varphi(\eta) = a - \frac{\beta g_w}{c_w}\eta + \left[\frac{B\beta g_w}{c_w} - \beta(1 - g_w)\, 6A\right]\frac{\eta^2}{2} + o(\eta^3) \qquad (49)$$

where

$$B = \frac{B'}{c_w} \qquad (50)$$

We now apply the boundary condition $f'(\infty) = 1$. Thus, taking only the first two terms in $f(\eta)$, we obtain:

$$\int_0^{\infty} \exp(B\eta)\exp(-A\eta^3)\left(a - \frac{\beta g_w}{c_w}\eta + \left[\frac{B\beta g_w}{c_w} - \beta(1 - g_w)\, 6A\right]\frac{\eta^2}{2}\right) d\eta = 1. \qquad (51)$$

This equation may be integrated in the form of gamma functions to give:

$$2c_w A^{4/3} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\Gamma\left(\frac{n+1}{3}\right) - A^{2/3}\left[1 + \beta(1 - g_w)\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\Gamma\left(\frac{n+3}{3}\right)\right] - \left[\frac{\beta g_w}{3c_w}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\Gamma\left(\frac{n+2}{3}\right) - \frac{\beta g_w}{6c_w}\sum_{n=0}^{\infty}\frac{\alpha^{n+1}}{(n+1)!}\Gamma\left(\frac{n+3}{3}\right)\right] = 0 \qquad (52)$$

where

$$\alpha = \frac{B}{A^{1/3}}. \qquad (53)$$

Taking equation (52) for A, we have

$$A = \left\{\frac{1 + \beta(1 - g_w)(1 + 0{\cdot}894\alpha) + \{[1 + \beta(1 - g_w)(1 + 0{\cdot}894\alpha)]^2 + 9{\cdot}65\beta g_w(1 + 0{\cdot}875\alpha)\}^{\frac{1}{2}}}{4c_w\,\Gamma(\frac{1}{3})(1 + 0{\cdot}51\alpha)}\right\}^{3/2} \qquad (54)$$

where we have taken terms to $o(a)$ only, i.e. we have assumed $a \ll 1$.

Returning to equation (49), it is clear that unless g_w is small the series diverges rapidly except for a low range of values of η. The quantity $\exp[-F(\eta)]\,\varphi(\eta)$ is of primary importance, and the value of its integral has been derived on the assumption that $\varphi(\eta)$ is a slowly varying function in comparison with $\exp[-F(\eta)]$. Thus, if $\varphi(\eta)$ becomes divergent for small values of η, the value of the integral determined by taking only three terms in $\varphi(\eta)$ will be considerably in error. The convergence of $\varphi(\eta)$ therefore has to be improved, and this is done by the use of Euler Transformations (see [3]). The series thus obtained was:

$$\varphi(\eta) = a - 0{\cdot}563\,\frac{\beta g_w}{c_w}\,\eta + \left[\frac{B\beta g_w}{c_w} - \beta(1 - g_w)\,6A\right]\frac{\eta^2}{32} + o(\eta^3) \tag{55}$$

The expression for A thus modified becomes:

$$A = \left\{\frac{1 + (\beta/16)(1 - g_w)(1 + 0{\cdot}894a) + [\{1 + (\beta/16)(1 - g_w)(1 + 0{\cdot}894a)\}^2 + 5{\cdot}45\beta g_w(1 + 1{\cdot}13a)]^{\frac{1}{2}}}{4c_w(1 + 0{\cdot}51a)\,\Gamma(\frac{1}{3})}\right\}^{3/2} \tag{56}$$

Solution of equation (28)

Let us now return to equation (28), the diffusion equation. This equation may be integrated to give:

$$K_1(\eta) = K_1'(o)\int_0^\eta \exp\left\{-\int_0^\eta[(S_w/c_w)f(\eta)\,d\eta]\right\}d\eta + K_{1w} \tag{57}$$

Again, we have taken the value of the flow parameters (P, c, S, etc.) to be that at the wall.

On substitution of the first three terms of $f(\eta)$, equation (57) becomes:

$$\frac{K_1(\eta)}{K_{2w}} = -B_1\int_0^\eta \{\exp(\beta_1\eta)\}\{\exp(-A_1\eta^3)\}\{\exp[(D_1\eta^4)/24]\}\,d\eta + \frac{K_{1w}}{K_{2w}} \tag{58}$$

where

$$D_1 = -\frac{S_w}{c_w}\left(Ba - \frac{\beta g_w}{c_w}\right),\; A_1 = S_wA,\; B_1 = S_wB$$

Thus, using the boundary conditions at infinity, we have

$$B_1\int_0^\infty \{\exp(B_1\eta)\}\{\exp(-A_1\eta^3)\}\{\exp[(D_1\eta^4)/24]\}\,d\eta = \frac{K_{1w}}{K_{2w}} \tag{59}$$

The left-hand side may be evaluated using gamma functions as before. Thus to $o(a)$ only, we obtain the following formula:

$$\frac{K_{1w}}{K_{2w}} = \frac{\Gamma(\frac{1}{3})}{3}\,a\,S_w^{2/3}\left\{1 + \frac{\Gamma(\frac{2}{3})\,S_w\,\beta g_w\,A^{-4/3}}{36\Gamma(\frac{1}{3})\,c_w^2}\right\} \tag{60}$$

It can be seen from equation (60) that unless β and g_w are very large, the terms involving β are small and may be neglected.

SKIN FRICTION AND SURFACE MASS TRANSFER

Now, from equations (41) and (56),

$$f''(o) = \frac{6\lambda^{3/2}(\beta, a)}{[4\Gamma(\frac{1}{3})]^{3/2}\, c_w^{\frac{1}{2}}} \tag{61}$$

Where

$$\lambda(\beta, a) = \left\{\frac{1 + (\beta/16)(1 - g_w)(1 + 0{\cdot}894a) + \{[1 + (\beta/16)(1 - g_w)(1 + 0{\cdot}894a)]^2 + 5{\cdot}45\beta g_w(1 + 1{\cdot}13a)\}^{\frac{1}{2}}}{(1 + 0{\cdot}51a)}\right\}$$

Also

$$K_1'(o) = -\frac{K_{2w}\, S_w\, a\lambda^{\frac{1}{2}}(\beta, a)}{[4\Gamma(\frac{1}{3})]^{\frac{1}{2}}\, c_w^{\frac{1}{2}}} \tag{62}$$

Define the surface mass-transfer parameter as

$$f_w = \frac{K_1'(o)\, c_w}{K_{2w}\, S_w} = -\, a c_w\, A^{1/3} \tag{63}$$

$$\text{i.e. } f_w = -\frac{a c_w^{\frac{1}{2}}\, \lambda^{\frac{1}{2}}(\beta, a)}{[4\Gamma(\frac{1}{3})]^{\frac{1}{2}}} \tag{64}$$

The skin-friction coefficient is defined by:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e\, u_e^2} \tag{65}$$

where

$$\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_w = \frac{\rho_w\, \mu_w\, u_e^2}{(2\xi)^{\frac{1}{2}}} f''(o) \tag{66}$$

Thus, we have

$$C_f = \frac{6(2)^{\frac{1}{2}}\, c_w^2\, A\mu_e}{(\xi)^{\frac{1}{2}}} \tag{67}$$

For the case $M_e = 0$, equation (67) becomes

$$C_f = \frac{6(2)^{\frac{1}{2}}\, c_w^2\, A}{(R_{ex})^{\frac{1}{2}}}(m + 1)^{\frac{1}{2}} \tag{68}$$

where

$$R_{ex} = \frac{\rho_e\, u_e x}{\mu_e}$$

and

$$m = \frac{\beta}{2 - \beta}$$

SURFACE HEAT TRANSFER

The heat transfer across the wall is made up of three components:

(a) Conduction heat
(b) Diffusion heat
(c) Radiation heat

The last of these three quantities will not be considered here. Therefore, considering only the first two, we may write the surface heat transfer as follows (see [3]):

$$q_w = -\frac{\mu_w}{P_w}\left[\left(\frac{\partial H}{\partial y}\right)_w - \sum_{i=1}^{2} h_i \left(\frac{\partial K_i}{\partial y}\right)_w + \frac{L_w h_{1w}}{K_{2w}}\left(\frac{\partial K_1}{\partial y}\right)_w\right] \tag{69}$$

whence we obtain:

$$q_w = -\frac{\rho_w \mu_w u_e H_e}{P_w (2\xi)^{\frac{1}{2}}}\left\{g'(o) + \frac{T_w C_{p1} K_1'(o)}{T_e C_{p2}[1 + (\gamma_e - 1)/2\, M_e^2]}\left[\frac{L_w}{K_{2w}} - \left(1 - \frac{C_{p2}}{C_{p1}}\right)\right]\right\} \tag{70}$$

which, for $M_e \to 0$, becomes

$$q_w = -\frac{c_w u_e H_e (m+1)^{\frac{1}{2}}}{P_w (R_{ex})^{\frac{1}{2}}}\left\{g'(o) + \frac{T_w C_{p1}}{T_e C_{p2}} K_1'(o)\left[\frac{L_w}{K_{2w}} - \left(1 - \frac{C_{p2}}{C_{p1}}\right)\right]\right\} \tag{71}$$

Thus, we may use equations (70) or (71) to determine the heat transfer to the surface. The quantities $K_1'(o)$ and K_{1w} have already been expressed as functions of the parameter α, and so it is now necessary to obtain a similar expression for $g_1'(o)$.

In order to do this, we refer back to the energy equation (27). This equation may be integrated to give:

$$\begin{aligned} g(\eta) = 1 - \int_\eta^\infty (P_w/c_w)\, \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\} \int_0^\eta \{\exp[\int_0^\eta (P_w/c_w) f\, d\eta]\}\, F^*\, d\eta\, d\eta \, - \\ \left\{\left[\int_\eta^\infty (P_w/c_w)\, \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\}\, d\eta\right] \Big/ \left[\int_0^\infty (P_w/c_w)\, \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\}\, d\eta\right]\right\} \times \\ \left\{1 - (H_w/H_e) - \int_0^\infty (P_w/c_w)\, \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\} \int_0^\eta \{\exp[\int_0^\eta (P_w/c_w) f\, d\eta]\}\, F^*\, d\eta\, d\eta\right\} \end{aligned} \tag{72}$$

where

$$F^* = \left[(1 - L_w)\frac{c_w}{P_w}\frac{(h_1 - h_2)}{H_e} K_1'(\eta)\right]' - \frac{u_e^2}{H_e}\left[\frac{c_w}{P_w}(P_w - 1) f' f''\right]' \tag{73}$$

Hence, we obtain:

$$\begin{aligned} g'(o) = \left\{1 - (H_w/H_e) - \int_0^\infty (P_w/c_w)\, \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\} \int_0^\eta \{\exp[\int_0^\eta (P_w/c_w) f\, d\eta]\}\, F^*\, d\eta\, d\eta\right\} \Big/ \\ \int_0^\infty \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\}\, d\eta \end{aligned} \tag{74}$$

As has already been done in previous sections, the flow parameters c, L, P, etc., are taken to have their wall values.

To evaluate $g'(o)$ we must integrate the expression:

$$J = \int_0^\infty (P_w/c_w)\, \{\exp[-\int_0^\eta (P_w/c_w) f\, d\eta]\} \int_0^\eta \{\exp[\int_0^\eta (P_w/c_w) f\, d\eta]\}\, F^*\, d\eta\, d\eta \tag{75}$$

This integration is carried out in two parts, corresponding to the two terms in equation (73). By suitable approximation, we may obtain:

$$J = \frac{u_e^2}{2H_e}(1-P_w)\{1-P_w[\tfrac{1}{2}+f''(o)f(o)]\}+(1-L_w)(h_1-h_2)_w\frac{K_1'(o)\,I_1}{H_e}$$
$$+\frac{u_e^2}{2H_e}(1-P_w)\frac{\beta g_w}{c_w}f(o)\,I_1+\frac{u_e^2}{2H_e}(1-P_w)\frac{\beta}{c_w}I_4 \qquad (76)$$

where

$$I_1 = \int_0^\infty \exp[-F(\eta)\,P_w]\,d\eta \qquad (77)$$

and

$$I_4 = \int_0^\infty \exp[-P_w F(\eta)]\int_0^\eta \exp[P_w F(\eta)]\,(d/d\eta)\,\{f(g-f'^2)\}\,d\eta\,d\eta \qquad (78)$$

Thus, we may obtain from equations (74) and (76),

$$g'(o) = I_1^{-1}\left\{1-\frac{H_w}{H_e}-\frac{u_e^2}{2H_e}(1-P_w)\{1-P_w[\tfrac{1}{2}+f''(o)f(o)]\}-(1-L_w)(h_1-h_2)_w\frac{K_1'(o)\,I_1}{H_e}\right.$$
$$\left.-\frac{u_e^2}{2H_e}(1-P_w)\left(\frac{\beta g_w}{c_w}f(o)\,I_1-\frac{\beta\,I_4}{c_w}\right)\right\} \qquad (79)$$

It may be noted that the pressure gradient parameter, β, assumes its maximum value in the stagnation region where $u_e \to 0$. In general, therefore, we will have two regions of flow, one in which the pressure gradient is large and Mach number small, and one in which the pressure gradient is small and Mach number large. In these regions the error caused by the neglect of the terms containing $\beta(u_e^2/H_e)$ should be minimized. There may be, however, a third region between these two in which both pressure gradient and Mach number are fairly large, and therefore these terms must be considered.

It now remains to evaluate the integral I_1.

Taking only the first three terms in $f(\eta)$, we obtain:

$$I_1 = \frac{1}{3A^{1/3}P_w^{1/3}}\sum_{n=0}^{\infty}\frac{a^n}{n!}P_w^{2n/3}\,\Gamma\left(\frac{n+1}{3}\right)+\frac{\beta g_w - B'a}{72\,c_w^2\,A^{5/3}P_w^{2/3}}\sum_{n=0}^{\infty}\frac{a^n}{n!}P_w^{2n/3}\,\Gamma\left(\frac{n+5}{3}\right) \qquad (80)$$

It is clear that the second group of terms only becomes of significant value when both the pressure gradient and injection rate are large. The magnitude of this term is also dependent upon the value of g_w, which becomes large when the injected gas has a high specific heat. Therefore, from equations (70), (79) and (80), it is possible to calculate the heat transfer at the plate surface.

STANTON NUMBER

It is usual to incorporate the heat-transfer rate into a non-dimensional coefficient termed the Stanton number, which is defined as follows:

$$C_H = \frac{q_w}{\rho_e\,u_e\,H_e\,[(H_w - H_{wAD})/H_e]} \qquad (81)$$

If we now consider the case when the heat-transfer rate is zero, i.e. adiabatic wall conditions, then from equations (70) and (74) we obtain:

$$1-\frac{H_{wAD}}{H_e}-J+\frac{T_{wAD}\,C_{p1}\,K_1'(o)\,I_1}{T_e\,C_{p2}\,\{1+[(\gamma_e-1)/2]\,M_e^2\}}\left\{\frac{L_w}{K_{2w}}-\left(1-\frac{C_{p2}}{C_{p1}}\right)\right\}=0 \qquad (82)$$

Assuming that

$$[K_1'(o)]_{AD}\, I_{1AD} = K_1'(o)\, I_1$$

But

$$\frac{H_w}{H_e} = \frac{T_w}{T_e}\,\frac{[K_{1w}\,(C_{p1}/C_{p2}) + K_{2w}]}{\{1 + [(\gamma_e - 1)/2]\, M_e^2\}} \tag{83}$$

and

$$\frac{H_{wAD}}{H_e} = \frac{T_{wAD}}{T_e}\,\frac{[K_{1w}\,(C_{p1}/C_{p2}) + K_{2w}]}{\{1 + [(\gamma_e - 1)/2]\, M_e^2\}} \tag{84}$$

From equations (82) and (84), we have

$$\frac{K_{1w}\,(C_{p1}/C_{p2}) + K_{2w}}{1 + [(\gamma_e - 1)/2]\, M_e^2} + \frac{T_e}{T_{wAD}}\, J_{wAD} - \frac{K_1'(o)\, I_1\,(C_{p1}/C_{p2})}{1 + [(\gamma_e - 1)/2]\, M_e^2}\left\{\frac{L_w}{K_{2w}} - [1 - (C_{p2}/C_{p1})]\right\} = \frac{T_e}{T_{wAD}} \tag{85}$$

where

$$J_{wAD} = \frac{u_e^2}{2H_e}\,(1 - P_w)\,[1 - P_w\,(\tfrac{1}{2} + f''(o)\, f(o))]_{wAD} + \frac{T_{wAD}}{T_e}\,\frac{[(C_{p1}/C_{p2}) - 1](1 - L_w)\, K_1'(o)\, I_1}{[1 + (\gamma_e - 1)/2\, M_e^2]} \tag{86†}$$

If we assume that $P_{wAD} = P_w$, $L_{wAD} = L_w$, $K_{1wAD} = K_{1w}$, then, from equation (85) we obtain:

$$\frac{T_{wAD}}{T_e} = \frac{\{1 - (u_e^2/2H_e)\,(1 - P_w)\,[1 - P_w\,(\tfrac{1}{2} + f''(o)\, f(o))]\}\,\{1 + (\gamma_e - 1)/2\, M_e^2\}}{K_{1w}\,(C_{p1}/C_{p2}) + K_{2w} + K_1'(o)\, I_1\,\{[(C_{p1}/C_{p2}) - 1]\,(1 - L_w) - (C_{p1}/C_{p2})\,[(L_w/K_{2w}) - 1 + (C_{p2}/C_{p1})]\}} \tag{87}$$

Thus, combining these results with equation (70) yields:

$$q_w = -\,\frac{\rho_w\,\mu_w\,u_e\,H_e}{P_w(2\xi)^{\frac{1}{2}}\, I_1}\left[1 - \frac{T_e\,T_w}{T_{wAD}\,T_e}\right]\left[1 - \frac{u_e^2}{2H_e}\,(1 - P_w)\{1 - P_w\,[\tfrac{1}{2} + f''(o)\, f(o)]\}\right] \tag{88}$$

here we have assumed that $[f''(o)]_{AD} = [f''(o)]$.

Hence, we have from equation (81),

$$C_H = \frac{C_w\mu_e\,(T_e/T_{wAD})}{P_w(2\xi)^{\frac{1}{2}}\, I_1}\,\frac{[1 - (u_e^2/2H_e)(1 - P_w)\,(1 - P_w\,\{\tfrac{1}{2} + f''(o)\, f(o)\})]}{K_{1w}\,(C_{p1}/C_{p2}) + K_{2w}} \tag{89}$$

Alternatively, from equations (82) and (76)

$$1 - \frac{u_e^2}{2H_e}\,(1 - P_w)\,\{1 - P_w\,[\tfrac{1}{2} + f''(o)\, f(o)]\} = \frac{H_{wAD}}{H_e} + \frac{(1 - L_w)\,(h_1 - h_2)_w}{H_e}\, K_1'(o)\, I_1 - \frac{h_{1wAD}}{H_e}\, K_1'(o)\, I_1\left[\frac{L_w}{K_{2w}} - 1 + \frac{C_{p2}}{C_{p1}}\right] \tag{90}$$

† Hereinafter, we shall omit the last term in the J expression in equation (76).

thus,

$$q_w = -\frac{c_w\,\rho_e\,\mu_e\,u_e\,H_e}{P_w(2\xi)^{\frac{1}{2}}\,I_1}\left(\frac{H_{wAD}-H_w}{H_e}\right)G \tag{91}$$

where

$$G = 1-(L_w-1)\,(h_1-h_2)_{wAD}\,\frac{K_1'(o)\,I_1}{H_e} - \frac{h_{1wAD}\,K_1'(o)\,I_1}{H_e}\left[\frac{L_w}{K_{2w}} - 1 + \frac{C_{p2}}{C_{p1}}\right] \tag{92}$$

From equations (81) and (91), we have

$$C_H = \frac{c_w \mu_e G}{P_w(2\xi)^{\frac{1}{2}}\,I_1} \tag{93}$$

From equations (67) and (93),

$$2P_w^{2/3}\,\frac{C_H}{C_f} = \frac{GA^{-2/3}}{2c_w \sum\limits_{n=0}^{\infty}(a^n/n!)\,P_w^{2n/3}\,\Gamma[(n+1)/3]} \tag{94}$$

where we have neglected the second group of terms in the I_1 expresion given by equation (80). Clearly, for $\beta = \alpha = 0$ this reduces to the modified Reynolds analogy. However, for any values of α and β other than zero, the right-hand side of equation (94) is not necessarily unity.

CONDUCTIVE HEAT TRANSFER

In certain cases it is of interest to consider only the conductive component of heat transfer. In this case, the heat transfer and Stanton number are defined as follows:

$$q_{wc} = -\frac{\rho_w\,\mu_w\,u_e\,H_e}{P_w(2\xi)^{\frac{1}{2}}}\left\{g'(o) + \frac{T_w\,C_{p1}\,K_1'(o)\,[(C_{p2}/C_{p1})-1]}{T_e\,C_{p2}\,\{1+[(\gamma_e-1)/2]\,M_e^2\}}\right\} \tag{95}$$

$$C_{Hc} = \frac{q_{wc}}{\rho_e\,u_e\,H_e\,[(T_w/T_e)-(T_{wADc}/T_e)]} \tag{96}$$

where $({}_{wADc})$ refers to conditions at the wall for zero conductive heat transfer.

Therefore, using equation (79) which applies whether we consider q_w or q_{wc}, we find from equation (95)

$$\frac{T_{wADc}}{T_e} = \frac{[1-(u_e^2/2H_e)\,(1-P_w)\,(1-P_w\{\frac{1}{2}+f''(o)\,f(o)\})]\{1+[(\gamma_e-1)/2]\,M_e^2\}}{K_{1w}\,(C_{p1}/C_{p2}) + K_{2w} + (2-L_w)\,[(C_{p1}/C_{p2})-1]\,K_1'(o)\,I_1} \tag{97}$$

Also, from physical considerations, for $M_e \to 0$,

$$\frac{T_{wADc}}{T_e} = 1. \tag{98}$$

First-order solutions for $M_e \to 0$

It is of interest to compare the value of the skin friction for a given blowing rate to the value for zero mass transfer. If we let C_{f0} be the value of the skin friction at zero mass transfer, then to $o(\alpha)$ only,

$$\frac{C_f}{C_{f0}} = \left(\frac{c_w}{c_{w0}}\right)^{\frac{1}{2}} (1 + 0{\cdot}51a)^{-3/2}$$
$$\left\{\frac{1 + (\bar{\beta}/16)(1 - g_w)(1 + 0{\cdot}894a) + [\{1 + (\bar{\beta}/16)(1 - g_w)(1 + 0{\cdot}894a)\}^2 + 5{\cdot}45\bar{\beta} g_w (1 + 1{\cdot}13a)]^{\frac{1}{2}}}{1 + (\bar{\beta}/16)(1 - g_{w0}) + [\{1 + (\bar{\beta}/16)(1 - g_{w0})\}^2 + 5{\cdot}45\bar{\beta} g_{w0}]^{\frac{1}{2}}}\right\}^{3/2} \tag{99}$$

where $g_{w0} = (T_w/T_e)$, and we have assumed $P_w = P_{w0}$

In this formula we may use the following approximate relation:

$$\frac{c_w}{c_{w0}} = \frac{\rho_w \mu_w}{(\rho_w \mu_w)_2} = \left[1 + K_{1w}\left(\frac{\mathcal{M}_2}{\mathcal{M}_1} - 1\right)\right]^{-\frac{1}{2}} \tag{100}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the molecular weights of the two gases. The value of K_{1w} is given by equation (60).

Similarly, taking terms to $o(a)$ only, and assuming $P_w = P_{w0}$,

$$\frac{C_{Hc}}{C_{Hc0}} = \left(\frac{c_w}{c_{w0}}\right)^{\frac{1}{2}} \frac{(1 + 0{\cdot}51\, a)^{-\frac{1}{2}}}{(1 + 0{\cdot}51\, P_w^{2/3}\, a)}$$
$$\left\{\frac{1 + (\bar{\beta}/16)(1 - g_w)(1 + 0{\cdot}89a) + [\{1 + (\bar{\beta}/16)(1 - g_w)(1 + 0{\cdot}89a)\}^2 + 5{\cdot}45\bar{\beta} g_w (1 + 1{\cdot}13a)]^{\frac{1}{2}}}{1 + (\bar{\beta}/16)(1 - g_{w0}) + [\{1 + (\bar{\beta}/16)(1 - g_{w0})\}^2 + 5{\cdot}45\bar{\beta} g_{w0}]^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \tag{101}$$

where we have neglected the second term in equation (80).

First-order solutions for finite M_e

It may be seen from equations (26) and (27) that unless the two quantities $\bar{\beta}[1 + \{(\gamma_e - 1)/2\}M_e^2]$ and (u_e^2/H_e) are both constant, conditions of similarity are violated.

The first order expression for the skin-friction coefficient ratio is given by exchanging β for $\bar{\beta}$ in equation (99).

We also obtain:

$$\frac{C_{Hc}}{C_{Hc0}} = \left\{K_{1w}\frac{C_{p1}}{C_{p2}} + K_{2w} + (2 - L_w)\left(\frac{C_{p1}}{C_{p2}} - 1\right) K_1'(o)\, I_1\right\} \left(\frac{C_{Hc}}{C_{Hc0}}\right)_{M_e=0} \tag{102}$$

DISCUSSION AND COMPARISON OF RESULTS

The expressions for the skin friction and heat-transfer coefficient ratios $[(C_f/C_{f0}), (C_{Hc}/C_{Hc0})]$ in equations (99) and (101) were plotted for various flow conditions and various injected gases. The first case to be considered was that of air injection under stagnation and flat plate conditions, where the external flow Mach number was assumed to be negligible. The present results were compared with the numerical data prepared by Baron [4] where a temperature ratio (T_w/T_e) was taken to be 0·8. It can be seen from Figs. 1(a), 1(b) that for $\bar{\beta} = 0$ and $\bar{\beta} = 1$ the results are in very close agreement throughout the range of $0 < f_w < 0{\cdot}5$. Outside this range of injection the parameter a becomes greater than unity, however the present solution still appears to be fairly accurate.

This first example indicates that the form of the solution is correct as far as the parameter a is concerned. The effect of the temperature ratio, (T_w/T_e), is of interest, and Fig. 2 has been prepared to accentuate this parameter. It can be seen that for $0 < (T_w/T_e) < 1$ the comparison with results from the numerical solution presented by Reshotko and Cohen [5] is very good; indicating that the form of the solution is also reasonable with respect to this variable.

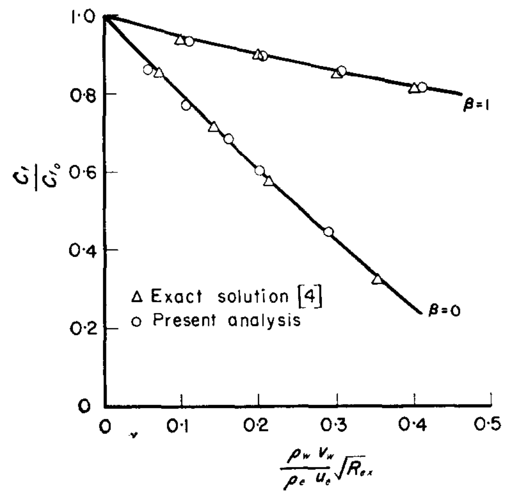

FIG. 1(a). Effect of air injection on skin-friction coefficient ($M_e = 0$, $T_w/T_e = 0{\cdot}8$).

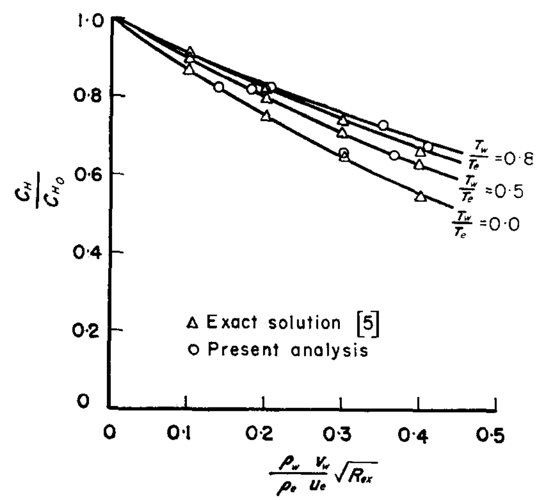

FIG. 2. Effect of T_w/T_e on heat-transfer coefficient ($M_e = 0$, air injection).

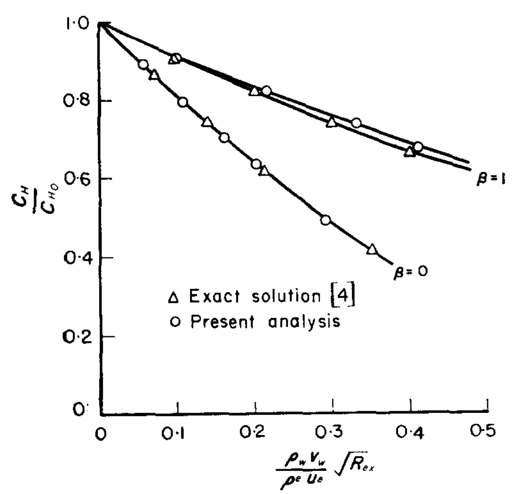

FIG. 1(b). Effect of air injection on heat-transfer coefficient ($M_e = 0$, $T_w/T_e = 0{\cdot}8$).

The third parameter which has a major effect on the solution for the case of air injection is the pressure gradient parameter, β. The previous analyses indicate that the role of this factor is correctly represented, but in order to fully investigate this point, the variation of $f''(o)$ with β for $(T_w/T_e) = 1$ was plotted on Fig. 3. These results are compared with numerical data from a report by Hartree [6], who carried out an analysis of a single component isothermal boundary layer. These results indicate that the effect of the pressure gradient is slightly underestimated with respect to the velocity gradient at the wall. The maximum error is about 8 per cent. It is clear, therefore, that while the combined effect of β and α appears to give accurate results, the individual effect of each is a little in error. The troublesome term in equation (56) is $A_4\beta(1 + B_4\alpha)$ where in the present analysis $A_4 = 5{\cdot}45$ and $B_4 = 1{\cdot}13$. It would seem that the value of A_4 is underestimated, while that of B_4 is overestimated.

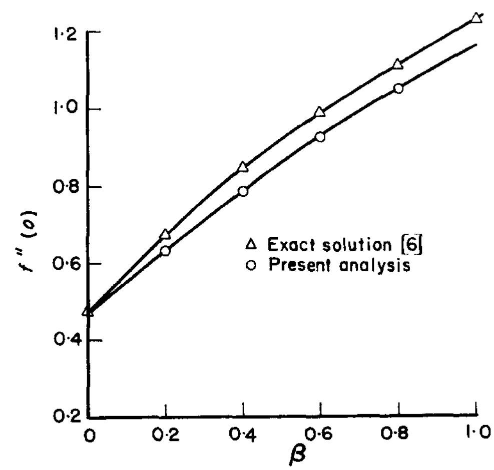

FIG. 3. Variation of $F''(o)$ with β (no injection).

However, under consideration of the rather large increase in $f''(o)$, the solution from this viewpoint appears to be satisfactory.

The results so far have demonstrated fairly convincingly the validity of the present solution when M_e approaches zero, it remains to investigate the effect of the Mach number on the solution. The results as predicted by equations (99) and (102) are shown in Fig. 4 in which we

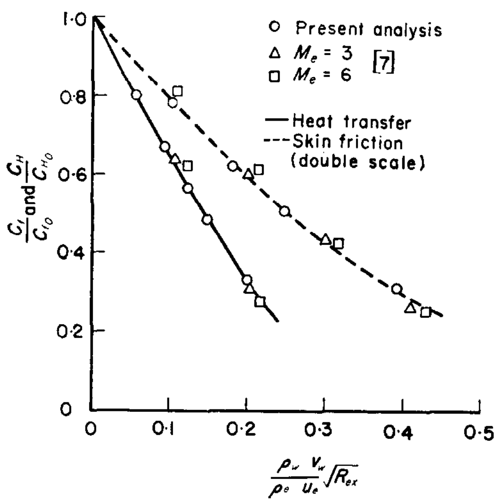

FIG. 4. Effect of Mach number on skin-friction and heat-transfer coefficients ($\beta = 0$, $T_w/T_e = 0{\cdot}8$, helium injection).

compare the present results with those from [7]. It may be seen that while the Mach number does appear to affect the solution, the variation over the Mach number range considered is small. It may be concluded that the predicted trend is acceptable for the case $\beta = 0$. The only remaining situation to be investigated for air injection is the combined effect of pressure gradient and Mach number, but since there appears to be no available exact data, this comparison cannot be made at the present time.

In the preceding sections the effect of air injection on the flow parameters has been investigated. However, it is of great importance to know the effect of injecting a foreign gas, since numerical solutions have shown this to be of particular interest. Two gases were chosen for which exact data was available, these were helium and hydrogen. For the former case the results were plotted in Figs. 5(a) and 5(b) for three values of the pressure gradient, while the temperature ratio was maintained at 0·8. The heat-transfer coefficient was calculated with

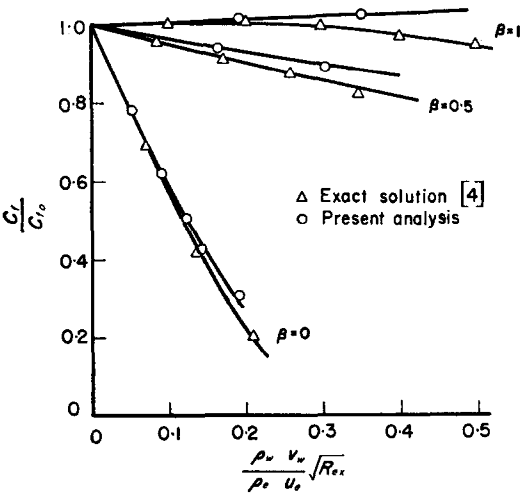

FIG. 5(a) Effect of helium injection on skin-friction coefficient ($M_e = 0$, $T_w/T_e = 0{\cdot}8$).

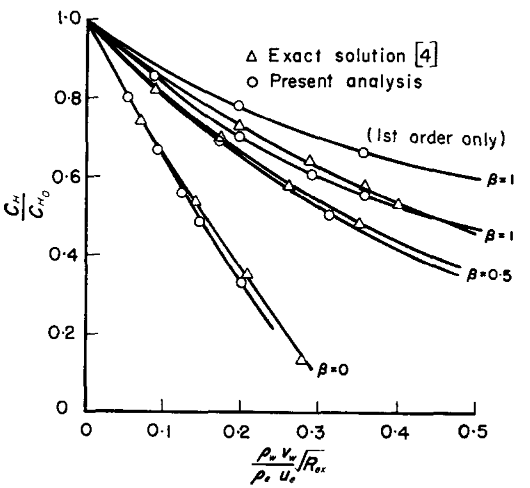

FIG. 5(b). Effect of helium injection on heat-transfer coefficient ($M_e = 0$, $T_w/T_e = 0{\cdot}8$).

the inclusion and omission of the second term in equation (80). It may be seen that the results for $\beta = 0$ are very good, but for $\beta = 1$, it is necessary to include the second term to obtain good accuracy. This necessity arises due to the increasing magnitude of g_w as the specific heat ratio, C_{p1}/C_{p2}, becomes large. However, taking this more exact form, the predicted reduction in the heat-transfer coefficient appears to be quite accurate for all three values of the pressure gradient. Figure 5(a) shows the variation of the skin-friction coefficient. It can be seen that while the values for $\beta = 0$ are very accurate, for $\beta = 1$ there is a discrepancy. In equation (101), we have the product of two functions, one which becomes very small for high α, and one which becomes very large for high α. Consequently, any inaccuracy in the method becomes magnified due to this combination.

A similar analysis was carried out for the case of hydrogen as the injected gas. The comments on this comparison are very similar to those for helium. It may be seen [Figs. 6(a) and 6(b)] that the same trends are exhibited

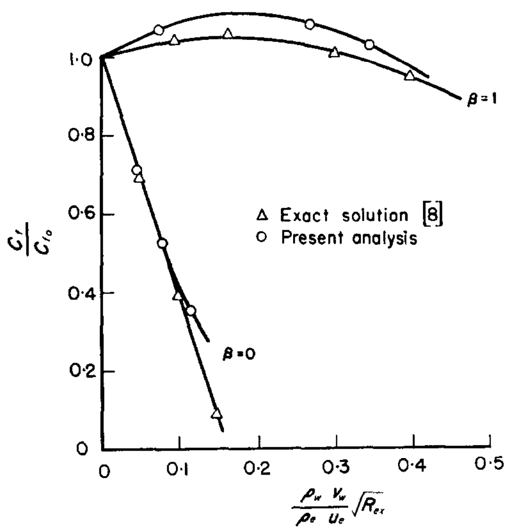

FIG. 6(a). Effect of hydrogen injection on skin-friction coefficient ($M_e = 0$, $T_w/T_e = 1{\cdot}0$).

but in a more accentuated manner. It is necessary to use the complete expression for I_1, in the evaluation of the heat-transfer coefficient, and also the over estimation of the skin friction is present. Probably the large magnitude of the molecular weight ratio ($\mathscr{M}_2/\mathscr{M}_1$) gives rise to the discrepancies. Figure 7 shows the effect of Mach number on the skin friction and heat-transfer coefficients. The comparison here is made with numerical data from [7].

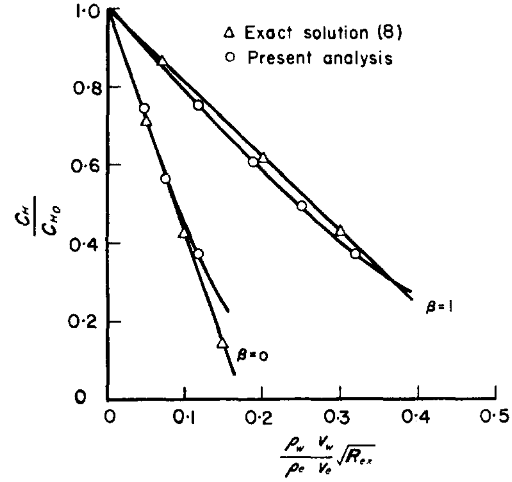

FIG. 6(b). Effect of hydrogen injection on heat-transfer coefficient ($M_e = 0$, $T_w/T_e = 1{\cdot}0$).

Figures 8 and 9 show the variation of wall mass friction with injection rate for both helium and hydrogen. While the values for zero pressure gradient are fairly close, there is a definite discrepancy for large values of β. It would appear that the present method underestimates the value of the wall mass fraction for high β.

This data for helium and hydrogen would suggest that the effect of foreign gas injection is reasonably accurately predicted, even for large molecular weight ratios, except in the prediction of the wall mass fraction for high α and high β.

CONCLUSIONS

The results would suggest that the approximate solutions are reasonably accurate over a wide range of flow conditions. There are several points of interest which emerge from the analysis, some of which have been shown by numerical analyses.

The results show quite clearly the large reduction of heat transfer which may be obtained as a result of injection at a relatively modest rate. Further, the very great improvement in the efficiency of the process by using a light injectant gas is also shown. However, there is seen to be a marked effect of the pressure gradient on this heat-transfer reduction process.

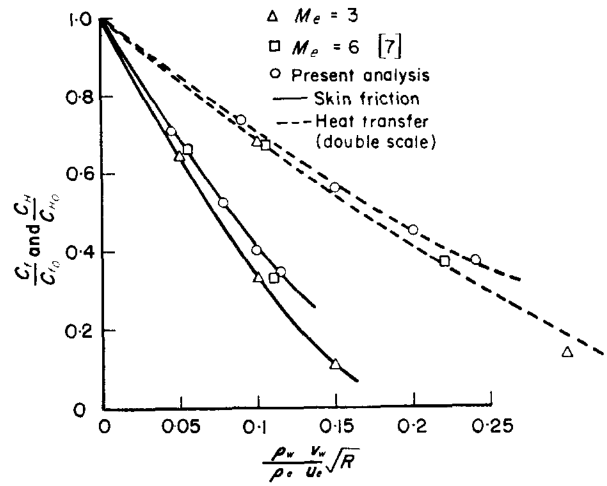

FIG. 7. Effect of Mach number on skin-friction and heat-transfer coefficients (hydrogen injection).

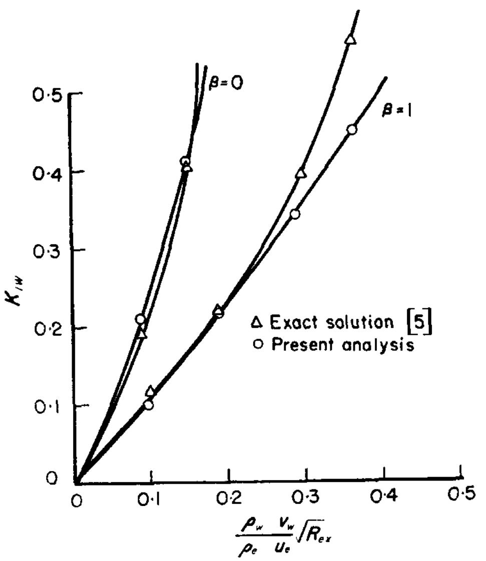

FIG. 8. Effect of injection on wall mass fraction (M_e = 0, T_w/T_e = 0·8, helium injection).

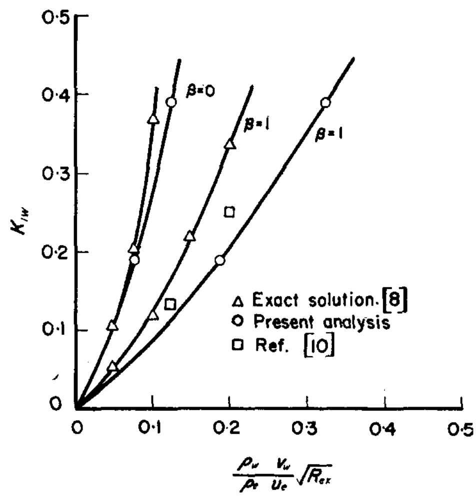

FIG. 9. Effect of injection on wall mass fraction (M_e = 0, T_w/T_e = 1·0, hydrogen injection).

The introduction of a large pressure gradient, such as would occur in the stagnation region of a body, greatly offsets the advantages of injection and returns the heat-transfer coefficient near to its solid wall value. Consequently, one would assume that heat protection by injection in the stagnation region would be of greatly reduced effect. However, this is not quite true if we take into consideration a further parameter, the wall temperature ratio, T_w/T_e. It is seen that as this quantity decreases, the adverse action of the pressure gradient decreases also. Thus, provided the wall temperature is maintained considerably lower than that of the external flow, the advantages of injection are utilized to their fullest extent.

This approximate method of solution has great applicability in producing optimum injection conditions when we are concerned with non-equilibrium flight conditions. Let us, for example, consider a body entering the atmosphere, and being subjected to deceleration over a time T. Let us assume that we wish to minimize the total heat transfer to the body over the flight time by the use of injection. It is apparent from the conclusions drawn earlier that the injection is more effective when the wall temperature ratio is lower. Consequently, it is unlikely that for a given amount of fluid available for injection purposes, the total heat flux will be a minimum if we eject this at a constant rate throughout the given time period. The quantity we wish to minimize is $\int_0^T q_w \, dt$ for a given value of $\int_0^T f_w \, dt$. This may be solved fairly easily by a numerical analysis of the various possibilities.

Perhaps one surprising fact which emerges from the results, and has also been shown by numerical procedures, is that the skin friction may be increased by the injection of a very light gas, such as hydrogen, while in the presence of a high pressure gradient. This is due to the relative effect of the decrease in viscosity and increase in velocity gradient at or near the surface, the latter occurring due to the greater acceleration of the low density gas in the high pressure gradient regions.

In conclusion, one may say that the approximate solutions appear to be reliable under most conditions which may be encountered in practice. The use of these formulae should be of great use in either solving simple problems quickly, or for analyzing more complex flight conditions where numerical procedures would be extremely unwieldly. Possible extensions of this work may be in the consideration of more complicated boundary layer conditions such as vectorial injection, or the inclusion of phenomena such as dissociation.

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Résumé—Cet article présente une analyse approchée des équations de la couche limite binaire lorsque le gradient de pression dans l'écoulement extérieur est différent de zéro et lorsque le nombre de Mach de l'écoulement n'est pas nécessairement faible. Des expressions pour les coefficients de frottement et de transport de chaleur sont obtenus en même temps que des formules montrant explicitement les effets de l'injection. Les résultats sont comparés avec les solutions numériques exactes dans une large gamme de conditions d'écoulement, et l'accord est très étroit.

Zusammenfassung—Diese Arbeit bringt eine Näherungsanalyse der Zweistoff-Grenzschichtgleichungen für Bedingungen, bei welchen der Druckgradient des äusseren Strömungsfeldes nicht Null und die Machzahl der Strömung nicht notwendig klein ist. Zusammen mit Formeln, welche die deutlichen Einflüsse der Querströmung aufzeigen, werden Ausdrücke für den Oberflächenreibungsbeiwert und die Wärmeübergangszahl abgeleitet. Die Ergebnisse werden mit genauen numerischen Lösungen für einen weiten Bereich von Strömungszuständen verglichen; die Übereinstimmung ist sehr gut.

Аннотация—В данной статье представлен приближенный анализ уравнений бинарного слоя в условиях, когда градиент давления поля во внешнем потоке не равен нулю, а число Маха для потока необязательно мало. Уравнения для коэффициентов трения и теплообмена выведены совместно с формулами, отражающими влияние вдува. Результаты сравнены с точными численными решениями для широкого диапазона условий потока. Они хорошо согласуются.